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ADDENDUM

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Abstract. Using the method of steepest descent, the author previously derived asymptotic expansions for the parabolic cylinder functions $D_p(z)$ (Crothers 1972), when $|z|^2$ and $|p|$ are large and of the same order. Some of the author's earlier remarks concerning the difficulties of the Green–Liouville method when $\arg(z) = \pm 3\pi/4$ are amplified.

Let me consider the papers of Olver (1959), Crothers (1972) and Lozano and Olver (1978). I quite agree with Lozano and Olver (1978) that *formally* one of the main results of Crothers (1972), namely (17), is immediately obtainable from Olver's equation (5.30*) by setting $\mu^2 = 1 + 2i\gamma$ and $\mu t 2^{1/2} = 2T_0\gamma^{1/2}\exp(3\pi i/4)$. It may well be possible, as Lozano and Olver (1978) suggest, to justify such a derivation via appropriate error analysis. However, in the absence of the latter I cannot agree that the formal derivation of Olver (1959) is in general justified, as I now argue.

Use of the exact recurrence relation

$$D_p(z) = \exp(i\pi p)D_p(-z) + ((2\pi)^{1/2}/\Gamma(-p))\exp[\tfrac{1}{2}i\pi(p+1)]D_{-p-1}(-iz) \tag{1}$$

together with Crothers (1972) equation (16) conjugated and his (13) yields in the notation of Crothers (1972).

$$\begin{aligned} D_{i\gamma}(\exp(3\pi i/4)2T_0\gamma^{1/2}) &\\ \simeq (\cos g\exp(-\tfrac{3}{4}\pi\gamma - i\gamma + i\gamma\ln\gamma + i\theta) + \sin g\exp(-\tfrac{3}{4}\pi\gamma - i\theta)) &\\ + (\sin g\exp(\tfrac{1}{4}\pi\gamma - i\theta) - \cos g\exp(-\tfrac{3}{4}\pi\gamma - i\gamma + i\gamma\ln\gamma + i\theta)) & \end{aligned} \tag{2}$$

The dilemma is now clear. The subdominant term of the second function exactly cancels the dominant term of the first function. To obtain the correct result given by Crothers (1972) equation (17), we must now argue empirically and neglect subdominant contributions, even though one of them is of the same order as (indeed apart from sign exactly equal to) one of the two dominant contributions. Such an empirical rule would clearly be most unsatisfactory by itself. Of course, this difficulty does not occur in the case of weak-coupling expansions, as I call them, namely when $|z| \gg \max(1, |p|)$, since for $\arg(z) = \pi/4$, $D_p(z)$ does not contain a subdominant term (cf Crothers 1972 equation (4)).

The persistent use of the above empirical procedure would constitute, in the words of Dingle (1973 p ix) 'failure to recognise that an unequivocal definition of an asymptotic expansion must be capable of *fixing* a given set of exponentially small terms, not just *permitting* those found by some procedure'. Nor is it beyond the bounds of imagination to question whether its application to functions, which are solutions of differential equations of order higher than the second, would be highly dangerous.

A careful reading of Olver (1954, 1956, 1958 and 1959) shows that the above dilemma is ignored in that the subdominant contributions such as $\exp(-\frac{7}{4}\pi\gamma - i\theta) \sin g$ and $\exp(-\frac{3}{4}\pi\gamma - i\gamma + i\gamma \ln \gamma + i\theta + i\pi) \cos g$ arising in (2) above are ignored. This is simply because Olver's (1959) asymptotic relation (4.3), which is derived from theorem A of Olver (1954, 1956, 1958), holds only in the sense of Poincaré. More specifically, this relation is established by matching up strong- and weak-coupling expansions for subdominant solutions on a Stokes' line, necessarily at large $|t|$ (that is, large $|z|/|p|^{1/2}$ in the current notation): a procedure which in other directions clearly neglects subdominant contributions to the strong-coupling expansions (that is, expansions which are valid for large $|p|$ and uniformly so with respect to z and $\arg p$). Nor should it be forgotten that subdominant contributions are only indeterminate on Stokes' lines in the presence of a dominant contribution.

Of course, it is not surprising that the above dilemma occurs in proceeding from (1) to (2), since each of the four terms should be multiplied by an inverse power series in γ, as given by Crothers (1972) equation (7). The advantage of the method of steepest descent, as used in Crothers (1972), is that the approximate use of exact recurrence relations and hence the dilemma are avoided. Indeed I submit that, provided the sum over l of Crothers (1972) (7) is included in each term, then the asymptotic expansions (13), (16), (17) and (18) of the same paper are exact, with no terms omitted or suppressed, precisely because, given that $\arg(z) = \pm\pi/4$, $\pm 3\pi/4$, the lines of steepest descent are parallel, as illustrated in figure 4 of Crothers (1972)†. (The philosophy behind such geometrical conveniences is well described by Dingle (1973 p 132)). I therefore also submit that the misconception, referred to in Lozano and Olver (1978), does not occur in Crothers (1972).

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† Note that the saddle points in figure 4 of Crothers (1972) are incorrectly labelled: they should be $[T_0 \pm (1 + T_0^2)^{1/2}]/2T_0$.